

Simultaneous separation for the Kowalevski and Goryachev–Chaplygin gyrostats

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2002 J. Phys. A: Math. Gen. 35 6419

(<http://iopscience.iop.org/0305-4470/35/30/315>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.107

The article was downloaded on 02/06/2010 at 10:17

Please note that [terms and conditions apply](#).

Simultaneous separation for the Kowalevski and Goryachev–Chaplygin gyrostats

Vadim B Kuznetsov

Department of Applied Maths, University of Leeds, Leeds LS2 9JT, UK

E-mail: vadim@maths.leeds.ac.uk

Received 11 January 2002

Published 19 July 2002

Online at stacks.iop.org/JPhysA/35/6419

Abstract

In the special case of zero square integral the Kowalevski gyrostat and Goryachev–Chaplygin gyrostat share a simple separation of variables originated from the 4×4 Lax matrix.

PACS numbers: 02.30.Ik, 02.30.Uu, 02.30.Zz, 02.40.Yy, 45.30.+s

Introduction

The theory of *separation of variables* (SoV) is the theory of special canonical transformations and the theory of *quantum separation of variables* is the theory of corresponding integral transforms. The former was already understood in this sense starting from the development of the Hamilton–Jacobi approach for solving Liouville integrable systems. The real power of the latter, which by now is undoubted, requires further investigation and demonstration.

The *speciality* of a separating transformation stems from its definition as a transform resulting in new variables *being separated* or equations *being decoupled* from one another. Below we give a (working) definition of SoV in the context of finite-dimensional integrable Hamiltonian dynamics.

By *separation of variables* for an n -degrees-of-freedom integrable system having n -independent Poisson commuting integrals $\{H_j(\mathbf{q}, \mathbf{p})\}_{j=1}^n$ we mean a canonical transformation from the (old) Darboux variables $q_j, p_j, j = 1, \dots, n$, to new Darboux variables $u_j, v_j, j = 1, \dots, n$, which satisfy the following *separation equations*:

$$\sum_{j=1}^n a_{ij}(u_i, v_i) H_j(\mathbf{q}, \mathbf{p}) = b_i(u_i, v_i) \quad i = 1, \dots, n. \quad (1)$$

In other words, being expressed in terms of the new variables, the integrals of motion H_j acquire the following ‘separated form’:

$$\mathbf{H} = \mathbf{A}^{-1} \mathbf{B} \quad (\mathbf{A})_{ij} = a_{ij} \quad (\mathbf{B})_i = b_i \quad (\mathbf{H})_i = H_i. \quad (2)$$

The conditions that the functions a_{ij} and b_i in (1) depend on the new (separation) variables with the index i only, is crucial. It indeed means that the n equations in (1) are really separated from one another. Therefore, they are n equations, each of one degree of freedom, sharing only the common values of the Hamiltonians H_j .¹

The above definition includes, as even more special canonical transform, the (classical) coordinate separation of variables when the new coordinates, say \mathbf{u} , are the functions of the old coordinates (\mathbf{q}) only, and they do not depend on the momenta (\mathbf{p}). See [18, 32, 33, 10] for some history of this sub-class of transformations and for many examples of such situation.

General separating canonical transforms, however, have new (separation) variables which are non-trivial functions of all $2n$ initial canonical variables,

$$u_j = u_j(\mathbf{q}, \mathbf{p}) \quad v_j = v_j(\mathbf{q}, \mathbf{p}) \quad j = 1, \dots, n \quad (3)$$

and which satisfy the standard Poisson brackets

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0 \quad j \neq k \quad \{v_j, u_j\} = 1 \quad j = 1, \dots, n. \quad (4)$$

Examples of such separating canonical transforms are usually much more sophisticated than those of the coordinate ones. As far as I know, the first explicit example was given by van Moerbeke in 1976 in [48] concerning the separation of variables for the periodic Toda lattice (see also [11]). In 1980 Kozlov [30] rewrote the classical results of Goryachev and Chaplygin on integration in quadratures of the Goryachev–Chaplygin top as a simple canonical transform. The method of SoV, viewed as a method of special canonical transformations, was further developed by Komarov in a series of works on tops, including quantum separation of variables, see [20–22]. Many further examples have been produced since 1982, with the theory benefiting mostly from the developments of the algebraic geometric and r -matrix understanding of the method of separation of variables. This led to a rather satisfying picture of the present state of the art of non-coordinate separation of variables. See [54, 16, 17] for more details.

Note here that in the realm of the algebraic geometry usually associated with many Liouville integrable systems, the separation equations (1) appear as equations for a set of separating Darboux coordinates on the spectral curve of a Lax matrix $L(u)$, namely:

$$(1) \Leftrightarrow \det(L(u_i) - v_i) = 0 \quad i = 1, \dots, n. \quad (5)$$

SoV is not unique, so that in the situation when there exist several Lax matrices for the same integrable system, one should expect several separations. But even for the same Lax matrix there are many different separations given by different sections², cf [36, 39, 43].

In the present paper we find a new canonical transform which simultaneously separates two classical tops: the Kowalevski top [29] and the Goryachev–Chaplygin top [13, 9]. The latter top is integrable only for the zero value of the square integral, $\ell = 0$ (which is one of the Casimirs), so that although the canonical transform will be defined for arbitrary value of ℓ , it will separate the Kowalevski top (and the other one) only for $\ell = 0$. SoV for the Kowalevski top for $\ell \neq 0$, which would generalize the found transform, remains an unsolved problem.

Sretenskii [55] discovered an integrable extension of the Goryachev–Chaplygin top adding the gyrostatic term to the Hamiltonian. Komarov [24] and, independently, Yehia [57] found the gyrostat extension of the Kowalevski top in 1987. The quantum Goryachev–Chaplygin gyrostat was treated in [23]. For quantum Kowalevski gyrostat see [24]. Here we will always include such gyrostatic terms (cf the parameter c below).

In section 1 we give definitions of the integrable systems concerned and their Lax matrices. In section 2 we recall the separation method with main results presented in section 3.

¹ As one can see, this definition is similar to the one defining Stäckel systems.

² Those usually differ by the number of non-moving poles of the Baker–Akhiezer function.

In section 4 we rewrite the found separating transformation through the generating function. The solution of the inverse problem is given in section 5. Finally, some concluding remarks can be found in the last section.

1. Algebra $e(3)$, tops and Lax matrices

The Poisson brackets for the $e(3)$ generators $J_k, x_k, k = 1, 2, 3$, are defined in the standard way:

$$\{J_k, J_l\} = \varepsilon_{klm} J_m \quad \{J_k, x_l\} = \varepsilon_{klm} x_m \quad \{x_k, x_l\} = 0 \tag{6}$$

where ε_{klm} is the completely anti-symmetric tensor, $\varepsilon_{123} = 1$.

The Casimirs of the bracket (6) have the form

$$C_1 = x_1 J_1 + x_2 J_2 + x_3 J_3 = \ell \quad C_2 = x_1^2 + x_2^2 + x_3^2 = 1. \tag{7}$$

The Kowalevski gyrostat has the Hamiltonian H ,

$$H = J^2 + (J_3 + c)^2 - 2bx_1 \tag{8}$$

and the second integral K ,

$$K = (J_3 + c)^2 J^2 + 2b(J_3 + c)(J_1 x_3 - J_3 x_1) - b^2 x_2^2 - 2b\ell J_1 \tag{9}$$

which are Poisson commuting:

$$\{H, K\} = 0. \tag{10}$$

Here we use the following notation for the square of the vector of angular momentum:

$$J^2 := J_1^2 + J_2^2 + J_3^2. \tag{11}$$

The two integrals of motion, H and K , define what is called Kowalevski gyrostat. It is a Liouville integrable system with two degrees of freedom.

Another integrable system closely related to the Kowalevski gyrostat is the Goryachev–Chaplygin gyrostat. It is integrable only when $\ell = 0$ and is defined by two Poisson commuting integrals of motion:

$$\hat{H} = J_1^2 + J_2^2 + (2J_3 + c)^2 - 4bx_1 \tag{12}$$

$$\hat{K} = (J_3 + c)(J_1^2 + J_2^2) + 2bx_3 J_1. \tag{13}$$

The Kowalevski top without the gyrostatic term ($c = 0$) was integrated by Kowalevski in 1889 [29]. See also [35] where a 2×2 Lax matrix was constructed for the Kowalevski top which is related to the separation of variables hidden in Kowalevski’s integration. There is a large body of literature dedicated to the Kowalevski top, including the study of its geometry. See [20, 24, 26, 14, 51, 5, 15, 8, 42, 28, 56, 47, 44, 25, 45, 46] to name just a few references. The Goryachev–Chaplygin top and gyrostat also received much attention in the literature starting from the discovery of this integrable top by Goryachev and Chaplygin in 1900. We refer the reader to the works [13, 9, 30, 21, 22, 23, 52, 6, 12, 7, 27].

In parallel with the notations J_1, J_2 and x_1, x_2 we will be using their equivalent complex versions J_+, J_- and x_+, x_- :

$$J_{\pm} = J_1 \pm iJ_2 \quad x_{\pm} = x_1 \pm ix_2 \tag{14}$$

with the $e(3)$ Poisson brackets (6) replaced, correspondingly, by

$$\{J_3, J_{\pm}\} = \mp iJ_{\pm} \quad \{J_+, J_-\} = -2iJ_3 \quad \{J_3, x_{\pm}\} = \{x_3, J_{\pm}\} = \mp ix_{\pm} \tag{15}$$

$$\{J_+, x_-\} = \{x_+, J_-\} = -2ix_3 \quad \{J_3, x_3\} = \{J_+, x_+\} = \{J_-, x_-\} = 0 \tag{16}$$

$$\{x_k, x_l\} = 0 \quad k, l = \pm, 3 \tag{17}$$

and the Casimirs (7) by

$$2C_1 = x_+J_- + x_-J_+ + 2x_3J_3 = 2\ell \quad C_2 = x_+x_- + x_3^2 = 1. \quad (18)$$

A 4×4 Lax matrix for the Kowalevski gyrostat was found in [51] and used in [8] to integrate the problem in terms of Prymian theta-functions. This integration is different from the one performed by Kowalevski. It is not obvious how to construct a separation of variables related to such Lax matrix. With the general case still being an interesting and challenging problem, we show here how to do it in the special case, when $\ell = 0$. That is, for this special case we construct a new separation of variables for the Kowalevski top (and simultaneously for the Goryachev–Chaplygin top) with separation variables belonging to the spectral curve of the 4×4 Lax matrix from [51].

The required Lax matrix is as follows:

$$L(u) = -i \begin{pmatrix} \frac{c}{u} & -\frac{bx_-}{u^2} & \frac{J_-}{u} & -\frac{bx_3}{u^2} \\ \frac{bx_+}{u^2} & -\frac{c}{u} & \frac{bx_3}{u^2} & -\frac{J_+}{u} \\ \frac{J_+}{u} & -\frac{bx_3}{u^2} & \frac{2J_3+c}{u} & 2 + \frac{bx_+}{u^2} \\ \frac{bx_3}{u^2} & -\frac{J_-}{u} & -2 - \frac{bx_-}{u^2} & -\frac{2J_3+c}{u} \end{pmatrix}. \quad (19)$$

The spectral curve $\Gamma: \det(L(u) - v) = 0$, of the Lax matrix (19) has the form

$$\Gamma: \left(v^2 + \frac{H}{u^2} - \frac{b^2}{u^4} \right)^2 - 4 \frac{u^4 v^2 + K}{u^4} - 4 \frac{c^2 u^4 - b^2 \ell^2}{u^6} = 0 \quad (20)$$

or

$$v^4 - 2v^2 \left(2 - \frac{H}{u^2} + \frac{b^2}{u^4} \right) - \frac{4c^2}{u^2} + \frac{H^2 - 4K}{u^4} + \frac{2b^2(2\ell^2 - H)}{u^6} + \frac{b^4}{u^8} = 0. \quad (21)$$

As I noted in [31] (cf also [7]) this Lax matrix contains the 3×3 Lax matrix $\hat{L}(u)$ for the Goryachev–Chaplygin top as its (1, 1)-minor:

$$\hat{L}(u) = -i \begin{pmatrix} -\frac{c}{u} & \frac{bx_3}{u^2} & -\frac{J_+}{u} \\ -\frac{bx_3}{u^2} & \frac{2J_3+c}{u} & 2 + \frac{bx_+}{u^2} \\ -\frac{J_-}{u} & -2 - \frac{bx_-}{u^2} & -\frac{2J_3+c}{u} \end{pmatrix}. \quad (22)$$

The spectral curve $\hat{\Gamma}: \det(\hat{L}(u) - v) = 0$, of the Lax matrix (22) has the form

$$v^3 - i \frac{cv^2}{u} - \left(4 - \frac{\hat{H}}{u^2} + \frac{b^2}{u^4} \right) v + \frac{4ic}{u} + i \frac{2\hat{K} - c\hat{H}}{u^3} + ib^2 \frac{c + 2x_3\ell}{u^5} = 0. \quad (23)$$

The Lax matrix (22) (and the curve (23)), for $\ell = 0$, was used in [7] to construct explicit theta-function formulae solving the dynamics of the Goryachev–Chaplygin top.

In the present paper I take one step further in studying a close connection between two problems and show that there exist separation variables $u_j, v_j, j = 1, 2$, which are canonical and which, for $\ell = 0$, belong to both curves, Γ and $\hat{\Gamma}$, simultaneously:

$$\begin{cases} \det(L(u_j) - v_j) = 0 \\ \det(\hat{L}(u_j) - v_j) = 0 \end{cases} \quad j = 1, 2 \quad (24)$$

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0 \quad j \neq k \quad \{v_j, u_j\} = 1 \quad j = 1, 2. \quad (25)$$

Therefore, I construct a new separation of variables which is characterized by the property of being a *simultaneous separation for both tops*.

2. The method of SoV

Separation variables had been generally used to construct closed expressions for the action variables (in terms of Abelian integrals) or to get a separated representation for the action function. Therefore, the SoV method had served for a long time an important but technical role in solving Liouville integrable systems of classical mechanics. A new and much more exciting application of the method came with the development of quantum integrable systems. Because of the fact that quantization of the action variables seemed to be a rather formidable task, quantum separation of variables became an inevitable refuge. In fact, it has been successfully performed for many families of integrable systems (see, for instance, survey [54]).

Starting from about 1982 the method of separation of variables gets connected with the R -matrix formalism of the quantum inverse scattering method, developed during that time by the Leningrad School. It was noted by Komarov (see [52, 54]) that for the 2×2 L -operators (Lax matrices) the separation variables ought to be the zeros of the off-diagonal element of the L -operator. This observation was fully exploited by Sklyanin in [52, 53] who developed a beautiful (pure algebraic) setting for the method within the framework of the R -matrix technique. Since then this approach took off and led to separations for many families of integrable systems. The method was further generalized to include higher rank L -operators and non-standard normalizations, see the 1995 review [54] and the later developments in [19, 37, 38, 34, 36, 39, 40, 41].

An alternative, algebraic geometric approach, which dates back to Adler and van Moerbeke [3, 4] and Mumford [49] and includes many researchers, have been developed starting from about the same time (see, for instance, [50, 15, 1, 2, 56, 16, 17]). It is based on thorough studies of the geometry of lower genus (algebraic completely) integrable systems. It has also led to many important new separations for complicated systems and tops.

Below we recall the most important formulae of the SoV method, adopting the algebraic description and following mainly the work [36] (see also [54, 34]).

A Baker–Akhiezer function f is the eigenvector of the Lax matrix for an integrable system

$$L(u)f = vf \quad (26)$$

considered as a function on the spectral curve $\Gamma : \det(L(u) - v) = 0$. The inverse scattering method pins down the separation variables as poles of this function (see the first example of this general fact in [48, 11]). There is, however, a large freedom of similarity transformations of the Lax matrix,

$$L(u) \mapsto VL(u)V^{-1} \quad (27)$$

which do not change the spectrum of $L(u)$ but change the divisor of poles of f . This freedom can be characterized, and therefore fixed, by introducing a *normalization* of the Baker–Akhiezer function,

$$\vec{\alpha} \cdot f \equiv \sum_{i=1}^N \alpha_i f_i = 1 \quad (f \equiv (f_1, \dots, f_N)^t) \quad (28)$$

which is given by a *normalization* (row-) vector $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$. In other words, one considers a *section of the line-bundle* (cf [16, 17, 43]). A proper (separating) normalization/section should give a divisor $\mathcal{D} = \sum_{j=1}^n (u_j, v_j)$ of moving poles of f , consisting of n (independent) points on the curve Γ whose coordinates are canonical variables. The canonicity of the separation variables is usually checked by a calculation involving an

r -matrix, but in lower genus situations it can be proved in a direct calculation. For a non-separating normalization/section the divisor \mathcal{D} will usually have more points than needed, which will not give canonical variables.

Many families of Lax matrices have a simple separating normalization when one of the components of the Baker–Akhiezer vector is put 1 (and, hence, one looks at the common poles of the other components), which corresponds to the following vector $\vec{\alpha}$:³

$$\vec{\alpha}_0 \equiv (1, 0, \dots, 0, 0). \quad (29)$$

We will call such normalization the *standard normalization*. There are examples of non-standard (dynamical) separating normalizations for the systems with elliptic r -matrix [54, 17], Calogero–Moser systems [36] and the D -type Toda lattice [34]. See also [43] where non-standard dynamical separating normalizations were used to construct Bäcklund transformations. Generally, Lax matrices with extra symmetries require non-standard separating normalizations.

Now, assuming that we know a separating normalization $\vec{\alpha}$ for an integrable system with the Lax matrix $L(u)$, let us derive the equations for the separation variables (u_j, v_j) , $j = 1, \dots, n$.

From the linear problem (26) and normalization (28) we derive that $\vec{\alpha} \cdot L^k f = v^k$, $k = 0, \dots, N - 1$, hence,

$$f = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha} \cdot L(u) \\ \vdots \\ \vec{\alpha} \cdot L^{N-1}(u) \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{N-1} \end{pmatrix}. \quad (30)$$

Another useful representation for the eigenvector f , which can be verified directly, is as follows:

$$f_j = \frac{(L(u) - v)_{jk}^{\wedge}}{(\vec{\alpha} \cdot (L(u) - v)_{\cdot k}^{\wedge})} \quad \forall k = 1, \dots, N \quad (31)$$

where the wedge denotes the adjoint matrix. It follows from this that the poles of f are the common zeros of the vector equation:

$$\vec{\alpha} \cdot (L(u) - v)^{\wedge} = 0. \quad (32)$$

Eliminating v from these equations, one can get a single equation for the u -components of the separating variables as zeros of the following determinant:

$$B(u) = \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha} \cdot L(u) \\ \vdots \\ \vec{\alpha} \cdot L^{N-1}(u) \end{pmatrix} = 0. \quad (33)$$

Note that this is exactly the denominator in the representation (30).

Also, from equations (32) we can obtain explicit formulae for the v -components of the separation variables in the form

$$v = A(u) \quad (34)$$

with $A(u)$ being rational functions of the entries of $L(u)$. Let us derive those formulae.

³ Or to any other similar vector with the 1 elsewhere.

Define the matrices $L^{(p)}$, $p = 1, \dots, N$, with the following entries:

$$L_{ij}^{(p)} := \sum_{i_1=1}^N \cdots \sum_{i_{p-1}=1}^N \begin{vmatrix} L_{i,j} & L_{i,i_1} & \cdots & L_{i,i_{p-1}} \\ L_{i_1,j} & L_{i_1,i_1} & \cdots & L_{i_1,i_{p-1}} \\ \vdots & \vdots & \ddots & \vdots \\ L_{i_{p-1},j} & L_{i_{p-1},i_1} & \cdots & L_{i_{p-1},i_{p-1}} \end{vmatrix} \quad p = 2, 3, \dots, N \quad (35)$$

and put $L^{(1)} \equiv L$. These matrices satisfy the recursion relation of the form

$$L^{(p)} = L(\text{tr}L^{(p-1)}) - (p - 1)L^{(p-1)}L. \quad (36)$$

Introduce the matrix $\mathcal{B}(u)$ by the formula

$$\mathcal{B}(u) := \begin{pmatrix} \vec{\alpha} \cdot L^{(1)}(u)L^{-1}(u) \\ \vec{\alpha} \cdot L^{(2)}(u)L^{-1}(u) \\ \frac{1}{2}\vec{\alpha} \cdot L^{(3)}(u)L^{-1}(u) \\ \dots \\ \frac{1}{(N-1)!}\vec{\alpha} \cdot L^{(N)}(u)L^{-1}(u) \end{pmatrix}. \quad (37)$$

With the help of this matrix we can represent the system of equations (32) as a system of linear homogeneous equations for the vector of powers of $(-v)$:

$$\vec{\alpha} \cdot (L(u) - v)^\wedge \equiv ((-v)^{N-1}, (-v)^{N-2}, \dots, 1) \cdot \mathcal{B}(u) = 0 \quad (38)$$

from which we derive that

$$(-v)^{j-i} = \frac{(\mathcal{B}^\wedge(u))_{ki}}{(\mathcal{B}^\wedge(u))_{kj}} \quad \forall k. \quad (39)$$

The formulae (39) give plenty of representations for the rational functions $A(u)$ in (34), all of them being compatible since, because of the equality

$$B(u) = (-1)^{[N/2]} \det(\mathcal{B}(u)) \quad (40)$$

the matrix $\mathcal{B}^\wedge(u_j)$ has rank 1. For more details see [36].

3. A new separation of variables

Consider the standard normalization vector

$$\alpha_0 = (1, 0, 0, 0) \quad (41)$$

for the 4×4 Lax matrix (19). Then, for $\ell = 0$, the defining equations,

$$(L(u) - v)^\wedge_{1k} = 0 \quad k = 1, 2, 3, 4 \quad (42)$$

give the following polynomial $B(u)$ for the separation variables u_1 and u_2 :

$$B(u) = u^4 + B_2u^2 + B_0 = (u^2 - u_1^2)(u^2 - u_2^2) \quad (43)$$

$$B_2 = \frac{2b(J_3 + c)(x_3J_- - x_-J_3) + b^2(x_-^2 + 2x_3^2)}{J_-^2 + 2bx_-} + \frac{2b^2x_3x_-(cJ_- + bx_3)}{J_-^2(J_-^2 + 2bx_-)} \quad (44)$$

$$B_0 = b^2 \frac{((J_3 + c)J_- + bx_3)^2}{J_-^2(J_-^2 + 2bx_-)}. \quad (45)$$

Note also the following useful formula for the variable B_2 :

$$B_2 = \frac{b^2x_-^2}{J_-^2 + 2bx_-} - \frac{2\sqrt{B_0}}{aJ_- \sqrt{J_-^2 + 2bx_-}}(x_-J_-J_3 - (J_-^2 + bx_-)x_3) \quad (46)$$

which is a linear expression in terms of J_3 and x_3 .

The corresponding rational function $A(u)$ can be chosen as follows:

$$A(u) = \frac{A_{-1}}{u} + \frac{A_{-3}}{u^3} \quad (47)$$

$$A_{-1} = ic + \frac{ix_3(J_-^2 + 2bx_-)}{x_-J_-} \quad A_{-3} = ib \frac{(J_3 + c)J_- + bx_3}{x_-J_-}. \quad (48)$$

Now, it is easy to check the following Poisson brackets between the two functions:

$$\{A(u), A(v)\} = \{B(u), B(v)\} = 0 \quad \forall u, v \in \mathbb{C}. \quad (49)$$

$$\{A(u), B(v)\} = \frac{2}{u^3(u^2 - v^2)}(u^4 B(v) - v^4 B(u))$$

Using these formulae one derives that the variables u_1, u_2 , defined by (43)–(46), and their conjugated counterparts,

$$v_j = A(u_j) \quad j = 1, 2 \quad (50)$$

are indeed canonical (Darboux) variables:

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0 \quad j \neq k \quad \{v_j, u_j\} = 1 \quad j = 1, 2. \quad (51)$$

These variables by construction satisfy (for $\ell = 0$) (42) and, therefore, (24).

Note here that the definitions of the separation variables do not depend on the value of ℓ and also that the brackets (51) are true for any ℓ .

4. Generating function

Let us fix a special representation of the underlying $e(3)$ algebra which will allow us to write the found canonical transformation explicitly through the generating function. It is a kind of holomorphic representation in terms of the Darboux variables $q_j, p_j, j = 1, 2$ ⁴

$$J_- = q_1 \quad x_- = q_2 \quad (52)$$

$$J_3 = i(q_1 p_1 + q_2 p_2) + \ell \quad x_3 = iq_2 p_1 + 1 \quad (53)$$

$$J_+ = q_1 p_1^2 + 2q_2 p_1 p_2 - 2i\ell p_1 - 2ip_2 \quad x_+ = q_2 p_1^2 - 2ip_1. \quad (54)$$

The meaning of this realization is in the fact that the variables J_- and x_- do not depend on the momenta \mathbf{p} and the variables J_3 and x_3 are linear in momenta. This together with linearity of the variables $u_1 u_2$ and $u_1^2 + u_2^2$ in terms of J_3 and x_3 makes it possible to integrate the equations explicitly in terms of elementary functions.

Finally, the canonical transformation defined in the previous section is given by the following generating function $F(\mathbf{u}|\mathbf{q})$:

$$F(\mathbf{u}|\mathbf{q}) = \frac{iq_1}{q_2} + \frac{ic}{2} \log(q_1^2 + 2bq_2) + i\ell \log(q_2) + \frac{ib^2 q_2}{2u_1 u_2 \sqrt{q_1^2 + 2bq_2}} + \frac{i}{2} \sqrt{q_1^2 + 2bq_2} \left(2 \frac{u_1 u_2}{b} - \frac{u_1^2 + u_2^2}{q_2 u_1 u_2} \right). \quad (55)$$

Therefore, the equations of the change of variables $(\mathbf{q}, \mathbf{p}) \leftrightarrow (\mathbf{u}, \mathbf{v})$ are written in the form

$$p_j = \frac{\partial F(\mathbf{u}|\mathbf{q})}{\partial q_j} \quad v_j = -\frac{\partial F(\mathbf{u}|\mathbf{q})}{\partial u_j} \quad j = 1, 2. \quad (56)$$

⁴ Cf a holomorphic representation of $sl(2)$.

5. Inverse problem

The inverse problem, i.e. finding expressions for the initial $e(3)$ variables in terms of separation variables, also has an explicit solution which is given below:

$$q_1^2 = -b^2 \frac{u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2)}{4u_1^6 u_2^6 (u_1 v_1 - u_2 v_2)^2} (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2)) + 4u_1^2 u_2^2 (u_1^2 - u_2^2) \quad (57)$$

$$J_- = q_1 \quad x_- = \frac{b (u_1^2 - u_2^2) (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2))}{2u_1^4 u_2^4 (u_1 v_1 - u_2 v_2)^2} \quad (58)$$

$$J_3 = \frac{1}{2u_1^2 u_2^2 (u_1 v_1 - u_2 v_2) (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2))} (ib^4 (u_1^2 - u_2^2)^2 + 2cu_1^4 u_2^4 (u_1 v_1 - u_2 v_2) (u_1^2 (v_1^2 - 2) - u_2^2 (v_2^2 - 2)) + 2cb^2 u_1^2 u_2^2 (u_1 v_1 - u_2 v_2) (u_1^2 - u_2^2) + 2ib^2 u_1^2 u_2^2 (u_1^2 v_1^2 - u_2^2 v_2^2) (u_1^2 - u_2^2) + iu_1^4 u_2^4 (u_1 v_1 - u_2 v_2) (v_1 u_1^3 (v_1^2 - 4) - v_2 u_2^3 (v_2^2 - 4)) + u_1 u_2 v_1 v_2 (u_1 v_1 - u_2 v_2)) \quad (59)$$

$$x_3 = \frac{2iq_1 u_1^2 u_2^2 (u_1^2 (u_1 v_1 - ic) - u_2^2 (u_2 v_2 - ic))}{b (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2))} \quad (60)$$

$$J_+ = -\frac{4q_1 u_1^4 u_2^4 (u_1 v_1 - u_2 v_2)}{b^2 (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2))^2} (u_1^3 u_2^3 v_1 v_2 (u_1 v_1 - u_2 v_2) + b^2 (u_1^3 v_1 - u_2^3 v_2) - icb^2 (u_1^2 - u_2^2) - icu_1^2 u_2^2 (u_1^2 v_1^2 - u_2^2 v_2^2) - c^2 u_1^2 u_2^2 (u_1 v_1 - u_2 v_2)) \quad (61)$$

$$x_+ = \frac{1}{b (u_1^2 (u_2^4 v_2^2 - b^2) - u_2^2 (u_1^4 v_1^2 - b^2))^2} (-4icb^2 u_1^2 u_2^2 (u_1^2 - u_2^2) (u_1^3 v_1 - u_2^3 v_2) + 2b^2 u_1^2 u_2^2 (u_1^2 - u_2^2) (u_1^4 v_1^2 - u_2^4 v_2^2) - 2b^2 c^2 u_1^2 u_2^2 (u_1^2 - u_2^2)^2 - 2c^2 u_1^4 u_2^4 (u_1^2 - u_2^2) (u_1^2 (v_1^2 - 4) - u_2^2 (v_2^2 - 4)) - 4icu_1^4 u_2^4 (u_1^3 v_1 - u_2^3 v_2) (u_1^2 (v_1^2 - 4) - u_2^2 (v_2^2 - 4)) + 2u_1^4 u_2^4 ((u_1^2 v_1^2 - u_2^2 v_2^2) (u_1^4 v_1^2 - u_2^4 v_2^2) - 4 (u_1^3 v_1 - u_2^3 v_2)^2)). \quad (62)$$

Here it is assumed that $\ell = 0$.

6. Concluding remarks

A separation of variables for the Kowalevski gyrostat, with $\ell = 0$, is found for the first time. It appeared to separate the Goryachev–Chaplygin gyrostat as well, thereby giving another separation for this problem.

If we put $c = 0$, i.e. switch off the gyrostatic term, we can compare our results with the original Kowalevski's and Goryachev–Chaplygin's separations for the respective tops

(in the $\ell = 0$ case). It is easy to see that the new separation is as complicated as the (simple) Goryachev–Chaplygin separation and it is *much simpler* than the Kowalevski separation.

Hence, the new separation stands a good chance to be quantized.

Acknowledgment

The support of the EPSRC is gratefully acknowledged.

References

- [1] Adams M R, Harnad J and Hurtubise J 1991 Coadjoint orbits, spectral curves and Darboux coordinates *The Geometry of Hamiltonian Systems (Berkeley, CA, 1989)* (New York: Springer) pp 9–21
- [2] Adams M R, Harnad J and Hurtubise J 1993 Darboux coordinates and Liouville–Arnol’d integration in loop algebras *Commun. Math. Phys.* **155** 385–413
- [3] Adler M and van Moerbeke P 1980 Completely integrable systems, Euclidean Lie algebras, and curves *Adv. Math.* **38** 267–317
- [4] Adler M and van Moerbeke P 1980 Linearization of Hamiltonian systems, Jacobi varieties and representation theory *Adv. Math.* **38** 318–79
- [5] Adler M and van Moerbeke P 1988 The Kowalevski and Hénon–Heiles motions as Manakov geodesic flows on $SO(4)$ —a two-dimensional family of Lax pairs *Commun. Math. Phys.* **113** 659–700
- [6] Bechlivanidis C and van Moerbeke P 1987 The Goryachev–Chaplygin top and the Toda lattice *Commun. Math. Phys.* **110** 317–24
- [7] Bobenko A I and Kuznetsov V B 1988 Lax representation and new formulae for the Goryachev–Chaplygin top *J. Phys. A: Math. Gen.* **21** 1999–2006
- [8] Bobenko A I, Reyman A G and Semenov-Tian-Shansky M A 1989 The Kowalevski top 99 years later: a Lax pair, generalizations and explicit solutions *Commun. Math. Phys.* **122** 321–54
- [9] Chaplygin S A 1901 Novyj sluchaj vrashcheniya tyazhelogo tverdogo tela, podpertogo v odnoj tochke. Trudy Otdeleniya Fizicheskikh Nauk Imperatorskogo Obshchestva Lyubitelej Estestvoznaniya X
See also Chaplygin S A 1933 *Collected Works* vol I (Leningrad, USSR: Academy of Sciences)
- [10] Eilbeck J C, Enol’skii V Z, Kuznetsov V B and Tsiganov A V 1994 Linear r-matrix algebra for classical separable systems *J. Phys. A: Math. Gen.* **27** 567–78
- [11] Flaschka H and McLaughlin D W 1976 Canonically conjugate variables for the Korteweg–de Vries equation and the Toda lattice with periodic boundary conditions *Prog. Theor. Phys.* **55** 438–56
- [12] Gavrilov L N 1987 Explicit solutions of the Gorjatchev–Tchaplygin top *C. R. Acad. Bulg. Sci.* **40** 19–22
- [13] Goryachev D N 1900 Novyj sluchaj razreshimosti zadachi o vrashchenii tyazhelogo tela s nepodvizhnoju tochkoju opory *Mat. Sb.* **21** 431–8
- [14] Haine L and Horozov E 1987 A Lax pair for Kowalevski’s top *Physica D* **29** 173–80
- [15] Horozov E and van Moerbeke P 1989 The full geometry of Kowalevski’s top and $(1, 2)$ -Abelian surfaces *Commun. Pure Appl. Math.* **42** 357–407
- [16] Hurtubise J C 1997 The algebraic geometry of the Kostant–Kirillov form *J. London Math. Soc.* **56** 504–18
- [17] Hurtubise J C and Kjiri M 2000 Separating coordinates for the generalized Hitchin systems and the classical r-matrices *Commun. Math. Phys.* **210** 521–40
- [18] Kalnins E G 1986 *Separation of Variables for Riemannian Spaces of Constant Curvature* (Harlow: Longman Scientific & Technical)
- [19] Kalnins E G, Kuznetsov V B and Miller W Jr 1995 Separation of variables and XXX Gaudin magnet *Rend. Sem. Mat. Univ. e Politec. Torino* **53** 109–20
- [20] Komarov I V 1981 The Kovalevskaja basis for the hydrogen atom *Teor. Mat. Fiz.* **47** 67–72
- [21] Komarov I V 1982 The Goryachev–Chaplygin top in quantum mechanics *Teor. Mat. Fiz.* **50** 402–09
- [22] Komarov I V 1982 Exact solution of the Goryachev–Chaplygin problem in quantum mechanics *J. Phys. A: Math. Gen.* **15** 1765–73
- [23] Komarov I V and Zalipaev V V 1984 The Goryachev–Chaplygin gyrostat in quantum mechanics *J. Phys. A: Math. Gen.* **17** 1479
- [24] Komarov I V 1987 A generalization of the Kovalevskaya top *Phys. Lett. A* **123** 14–5
- [25] Komarov I V 2001 Remarks on Kowalevski’s top *J. Phys. A: Math. Gen.* **34** 2111–20 *Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000)*

- [26] Komarov I V and Kuznetsov V B 1987 Quasiclassical quantization of the Kovalevskaya top *Teor. Mat. Fiz.* **73** 335–47 (Engl. transl. 1987 *Theor. Math. Phys.* **73** 1255–63)
- [27] Komarov I V and Kuznetsov V B 1989 Generalization of the Goryachev–Chaplygin gyrostat in quantum mechanics *J. Sov. Math.* **47** 2459–66
- [28] Komarov I V and Kuznetsov V B 1990 Kowalevski’s top on the Lie algebras $\mathfrak{o}(4)$, $\mathfrak{e}(3)$ and $\mathfrak{o}(3,1)$ *J. Phys. A: Math. Gen.* **23** 841–6
- [29] Kowalevski S 1889 Sur le problème de la rotation d’un corps solide autour d’un point fixe *Acta Math.* **12** 177–232
- [30] Kozlov V V 1980 *Metody kachestvennogo analiza v dinamike tverdogo tela* (in Russian) *Methods of Qualitative Analysis in the Dynamics of a Rigid Body* (Moscow: Moskov. Gos. Univ.) p 231
- [31] Kuznetsov V B 1989 Applications of the inverse scattering method to 2-dimensional classical and quantum integrable systems *PhD Thesis* Leningrad University p 118
- [32] Kuznetsov V B 1992a Quadrics on real Riemannian spaces of constant curvature: separation of variables and connection with Gaudin magnet *J. Math. Phys.* **33** 3240–54
- [33] Kuznetsov V B 1992b Equivalence of two graphical calculi *J. Phys. A: Math. Gen.* **25** 6005–26
- [34] Kuznetsov V B 1997 Separation of variables for the D_n -type periodic Toda lattice *J. Phys. A: Math. Gen.* **30** 2127–38
- [35] Kuznetsov V B 2002 Kowalevski top revisited *Surveys from Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Kowalevski property 2002 CRM Proc. and Lecture Notes (Leeds, April 2000))* ed V B Kuznetsov (Providence, RI: American Mathematical Society)
- [36] Kuznetsov V B, Nijhoff F W and Sklyanin E K 1997 Separation of variables for the Ruijsenaars system *Commun. Math. Phys.* **189** 855–77
- [37] Kuznetsov V B and Sklyanin E K 1995 Separation of variables in A_2 type Jack polynomials *RIMS Kokyuroku* **919** 27–34
- [38] Kuznetsov V B and Sklyanin E K 1996 Separation of variables for the A_2 Ruijsenaars system and a new integral representation for the A_2 Macdonald polynomials *J. Phys. A: Math. Gen.* **29** 2779–804
- [39] Kuznetsov V B and Sklyanin E K 1998 On Bäcklund transformations for many-body systems *J. Phys. A: Math. Gen.* **31** 2241–51
- [40] Kuznetsov V B and Sklyanin E K 1999 Separation of variables and integral relations for special functions *The Ramanujan Journal* **3** 5–35
- [41] Kuznetsov V B and Sklyanin E K 1999 Factorisation of Macdonald polynomials *Symmetries and Integrability of Difference Equations (London Mathematical Society, Lecture Note Series, vol 255)* ed P A Clarkson and F W Nijhoff (Cambridge: Cambridge University Press) pp 370–84
- [42] Kuznetsov V B and Tsiganov A V 1989 A special case of Neumann’s system and the Kowalevski–Chaplygin–Goryachev top *J. Phys. A: Math. Gen.* **22** L73–L79
- [43] Kuznetsov V B and Vanhaecke P 2002 Bäcklund transformations for finite-dimensional integrable systems: a geometric approach *Geom. Phys.* at press (ArXiv:nlin. SI 0004003)
- [44] Leprevost F and Markushevich D 1999 A tower of genus two curves related to the Kowalevski top *J. Reine Angew. Math.* **514** 103–11
- [45] Markushevich D 2001 Kowalevski top and genus-2 curves *J. Phys. A: Math. Gen.* **34** 2125–35
- [46] Markushevich D 2002 Some algebro-geometric integrable systems versus classical ones *Surveys from Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Kowalevski Property 2002 CRM Proc. and Lecture Notes (Leeds, April 2000))* ed V B Kuznetsov (Providence RI: American Mathematical Society)
- [47] Marshall I D 1998 The Kowalevski top: its r-matrix interpretation and bihamiltonian formulation *Commun. Math. Phys.* **191** 723–34
- [48] van Moerbeke P 1976 The spectrum of Jacobi matrices *Invent. Math.* **37** 45–81
- [49] Mumford D 1984 Jacobian theta functions and differential equations *Tata Lectures on Theta. II.* (Boston, MA: Birkhäuser Boston Inc.) With the collaboration of C Musili, M Nori, E Previato, M Stillman and H Umemura
- [50] Previato E 1987 Flows on r-gonal Jacobians *The Legacy of Sonya Kovalevskaya (Cambridge, MA and Amherst, MA, 1985)* (Providence, RI: American Mathematical Society) pp 153–80
- [51] Reyman A G and Semenov-Tian-Shansky M A 1987 Lax representation with a spectral parameter for the Kowalevski top and its generalizations *Lett. Math. Phys.* **14** 55–61
- [52] Sklyanin E K 1984 The Goryachev–Chaplygin top and the method of the inverse scattering problem *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **133** 236–257 (Differential geometry, Lie groups and mechanics, VI)
- [53] Sklyanin E K 1985 The quantum Toda chain *Nonlinear Equations in Classical and Quantum Field Theory (Meudon/Paris, 1983/1984)* (Berlin: Springer) pp 196–233

-
- [54] Sklyanin E K 1995 Separation of variables—new trends *Prog. Theor. Phys. Suppl.* **118** 35–60 *Quantum Field Theory, Integrable Models and Beyond (Kyoto, 1994)*
- [55] Sretenskii L N 1963 O nekotorykh sluchajakh integriruемости uravnenii dvizhenija gyrostata *DAN SSSR* **149** 292–4
Sretenskii L N 1963 *On certain cases of motion of a heavy rigid body with a gyroscope Vestn. Moskov. Univ. Ser. I Mat. Mekh.* **1963** 60–71
- [56] Vanhaecke P 1996 *Integrable Systems in the Realm of Algebraic Geometry* (Berlin: Springer)
- [57] Yehia H M 1987 *Novije integriruemije sluchai zadachi o dvizhenii gyrostata Vestn. MGU, Ser. Mat. Mekh.* **N4** pp 88–90